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## EXACTLY SOLVABLE PAIRING MODEL USING AN EXTENSION OF RICHARDSON-GAUDIN APPROACH

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We introduce a new class of exactly solvable boson pairing models using the technique of Richardson and Gaudin. Analytical expressions for all energy eigenvalues and first few energy eigenstates are given. In addition, another solution to Gaudin's equation is also mentioned. A relation with the Calogero-Sutherland model is suggested.

### 1. Introduction

The notion of pairing plays a central role in the BCS theory of superconductivity.<sup>1</sup> In 1963, R.W. Richardson showed that exact energy eigenvalues and eigenstates of the BCS pairing Hamiltonian can be computed if one can solve a given set of highly coupled nonlinear equations.<sup>2,3</sup> The limit of these equations in which the BCS coupling constant is very large were also obtained by Gaudin in 1976. Gaudin's tool was the algebraic Bethe ansatz method.<sup>4,5</sup> Gaudin's Hamiltonians are the constants of motion of BCS Hamiltonian in the large coupling constant limit.

In this paper, we present a new class of exactly solvable boson pairing models using the technique pioneered by Richardson and Gaudin. The technique is outlined in the next section and in Section 3 we introduce the new class of exactly solvable models together with all energy eigenvalues and first few energy eigenstates.

Richardson-Gaudin technique is based on an equation of which three solutions are known. To each solution there is a corresponding set of mutually commuting Hamiltonians and a related Lie algebra. In Section 4 we present another solution

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to Gaudin's equation. We identify the corresponding set of mutually commuting Hamiltonians together with the related Lie algebra.

## 2. Gaudin Algebra

Gaudin starts with the following operators<sup>4</sup>:

$$h_i^{(0)} = \sum_{\substack{j=1 \\ j \neq i}}^N \sum_{\alpha=1}^3 w_{ij}^{\alpha} t_i^{\alpha} t_j^{\alpha}. \quad (1)$$

Here  $w_{ij}^{\alpha}$  are arbitrary complex numbers and  $t_i^{\alpha}$  satisfy the  $SU(1,1)$  commutators:

$$[t_i^+, t_j^-] = -2\delta_{ij} t_j^0, \quad [t_i^0, t_j^{\pm}] = \pm \delta_{ij} t_j^{\pm}. \quad (2)$$

Then he shows that if we require these Hamiltonians to commute

$$[h_i^{(0)}, h_j^{(0)}] = 0 \quad (3)$$

the coefficients  $w_{ij}^{\alpha}$  must obey the following equation:

$$w_{ij}^{\alpha} w_{jk}^{\gamma} + w_{ji}^{\beta} w_{ik}^{\gamma} - w_{ik}^{\alpha} w_{jk}^{\beta} = 0. \quad (4)$$

Assuming **i)** the coefficients are antisymmetric under the exchange of  $i$  and  $j$

$$w_{ij}^{\alpha} + w_{ji}^{\alpha} = 0 \quad (5)$$

**ii)** the coefficient  $w_{ij}^{\alpha}$  can be expressed as a function of the difference between two real parameters  $u_i$  and  $u_j$ , and **iii)** the operator  $\sum_i t_i^0$  commutes with  $h_j^{(0)}$  for all  $j$ , Gaudin found three solutions to Eq. (4). These solutions are given by

$$w_{ij}^{\alpha} = \frac{1}{u_i - u_j} \quad \text{for } \alpha = 1, 2, 3, \quad (6)$$

$$w_{ij}^0 = p \cot[p(u_i - u_j)] \quad w_{ij}^{1,2} = \frac{p}{\sin[p(u_i - u_j)]}, \quad (7)$$

$$w_{ij}^0 = p \coth[p(u_i - u_j)] \quad w_{ij}^{1,2} = \frac{p}{\sinh[p(u_i - u_j)]}. \quad (8)$$

The operators which are obtained by substituting these solutions in Eq. (1) are referred to as rational, trigonometric and hyperbolic Gaudin magnet Hamiltonians, respectively.<sup>5,6</sup> In the rest of this section we consider the rational Hamiltonian

$$h_i^{(0)} = \sum_{\substack{j=1 \\ j \neq i}}^N \frac{\vec{t}_i \cdot \vec{t}_j}{u_i - u_j}. \quad (9)$$

The associated algebra, rational Gaudin algebra is an infinite dimensional complex Lie algebra generated by a one-parameter set of operators  $S^+(\lambda)$ ,  $S^-(\lambda)$  and  $S^0(\lambda)$ . Commutators for rational Gaudin algebra are given as follows:

$$[S^+(\lambda), S^-(\mu)] = -2 \frac{S^0(\lambda) - S^0(\mu)}{\lambda - \mu} \quad (10)$$

$$[S^0(\lambda), S^\pm(\mu)] = \pm \frac{S^\pm(\lambda) - S^\pm(\mu)}{\lambda - \mu} \quad (11)$$

$$[S^0(\lambda), S^0(\mu)] = [S^\pm(\lambda), S^\pm(\mu)] = 0 \quad (12)$$

Commutators of these generators for  $\lambda = \mu$  are found by taking the limit of the right hand sides of the above commutators as  $\lambda \rightarrow \mu$ .

Let us define the operator

$$H(\lambda) = S^0(\lambda)S^0(\lambda) - \frac{1}{2}S^+(\lambda)S^-(\lambda) - \frac{1}{2}S^-(\lambda)S^+(\lambda). \quad (13)$$

Using the commutators (10-12), it is easy to show that  $H(\lambda)$  forms a one parameter family of mutually commuting operators:

$$[H(\lambda), H(\mu)] = 0. \quad (14)$$

One can diagonalize these operators simultaneously, starting from a lowest weight vector and using  $S^+(\lambda)$  as step operators. Lowest weight vector  $|0\rangle$  by satisfies

$$S^-(\lambda)|0\rangle = 0, \quad \text{and} \quad S^0(\lambda)|0\rangle = W(\lambda)|0\rangle \quad (15)$$

for every  $\lambda \in \mathbb{C}$ . Here  $W(\lambda)$  is a complex function. One gets

$$H(\lambda)|0\rangle = E_0(\lambda)|0\rangle, \quad E_0(\lambda) = W(\lambda)^2 - W'(\lambda). \quad (16)$$

We write the Bethe ansatz state

$$|\xi_1, \xi_2, \dots, \xi_n\rangle \equiv S^+(\xi_1)S^+(\xi_2) \dots S^+(\xi_n)|0\rangle \quad (17)$$

for  $n$  arbitrary complex numbers  $\xi_1, \xi_2, \dots, \xi_n \in \mathbb{C}$ . This is an eigenvector of  $H(\lambda)$  if these complex numbers satisfy the following set of  $n$  Bethe ansatz equations<sup>5,6</sup>:

$$W(\xi_\alpha) = \sum_{\beta=1, \beta \neq \alpha}^n \frac{1}{\xi_\alpha - \xi_\beta} \quad \text{for} \quad \alpha = 1, 2, \dots, n. \quad (18)$$

If  $\xi_1, \xi_2, \dots, \xi_n \in \mathbb{C}$  is a solution of above equations then (17) is an eigenvector of  $H(\lambda)$  with the eigenvalue

$$E_n(\lambda) = E_0(\lambda) - 2 \sum_{\alpha=1}^n \frac{W(\lambda) - W(\xi_\alpha)}{\lambda - \xi_\alpha}. \quad (19)$$

There exists a realization of the rational Gaudin algebra in terms of the  $SU(1,1)$  generators of Eq. (2), given by

$$S^0(\lambda) = \sum_{i=1}^N \frac{t_i^0}{u_i - \lambda} \quad \text{and} \quad S^\pm(\lambda) = \sum_{i=1}^N \frac{t_i^\pm}{u_i - \lambda}. \quad (20)$$

Here  $u_1, u_2, \dots, u_N$  are arbitrary real numbers which are all different from each other and  $N \geq 0$ . In this realization  $H(\lambda)$  of Eq. (13) is given by

$$H(\lambda) = \sum_{i,j=1}^N \frac{\vec{t}_i \cdot \vec{t}_j}{(u_i - \lambda)(u_j - \lambda)}. \quad (21)$$

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$H(\lambda)$  has simple poles on the real axis. Residues of  $-H(\lambda)/2$  at the points  $\lambda = u_i$  are the rational Gaudin magnet Hamiltonians given in Eq. (9). Eq. (14) implies

$$[H(\lambda), h_i^{(0)}] = 0, \quad \text{and} \quad [h_i^{(0)}, h_j^{(0)}] = 0. \quad (22)$$

### 3. A Model of Interacting Bosons

The rational Gaudin algebra (10-12) can be realized in terms of the boson operators:

$$S^+(\lambda) = \frac{1}{2} \sum_{\beta=1}^{n_1} \Lambda_\beta \frac{b_\beta^\dagger b_\beta^\dagger}{x_1 - \lambda} + \frac{1}{2} \sum_{\alpha=1}^{n_2} \Xi_\alpha \frac{a_\alpha^\dagger a_\alpha^\dagger}{x_2 - \lambda} \quad (23)$$

$$S^-(\lambda) = \frac{1}{2} \sum_{\beta=1}^{n_1} \Lambda_\beta \frac{b_\beta b_\beta}{x_1 - \lambda} + \frac{1}{2} \sum_{\alpha=1}^{n_2} \Xi_\alpha \frac{a_\alpha a_\alpha}{x_2 - \lambda} \quad (24)$$

$$S^0(\lambda) = \frac{1}{4} \sum_{\beta=1}^{n_1} \frac{b_\beta^\dagger b_\beta + b_\beta b_\beta^\dagger}{x_1 - \lambda} + \frac{1}{4} \sum_{\alpha=1}^{n_2} \frac{a_\alpha^\dagger a_\alpha + a_\alpha a_\alpha^\dagger}{x_2 - \lambda} \quad (25)$$

Here  $x_1$  and  $x_2$  are two real parameters which differ from each other and  $\Lambda_\mu^2 = \Xi_\alpha^2 = 1$ . There are two types of bosons named  $b_\beta$  and  $a_\alpha$  for  $\beta = 1, 2, \dots, n_1$  and  $\alpha = 1, 2, \dots, n_2$ . The operators  $b_\beta, b_\beta^\dagger$  and  $a_\alpha, a_\alpha^\dagger$  create or annihilate these bosons:

$$[b_\beta, b_\gamma^\dagger] = \delta_{\beta\gamma} \quad [a_\alpha, a_\sigma^\dagger] = \delta_{\alpha\sigma}. \quad (26)$$

If we choose  $n_1 = 5$ ,  $n_2 = 1$ ,  $\Lambda_\beta = 1$ , and,  $\Xi_\alpha = -1$  then above operators can be considered as a generalization of the *sd*-version of the interacting boson model.  
7,8,9,10

The lowest weight state is the boson vacuum and corresponding  $W(\lambda)$  is

$$W(\lambda) = \frac{1}{4} \left( \frac{n_1}{x_1 - \lambda} + \frac{n_2}{x_2 - \lambda} \right). \quad (27)$$

In this realization,  $H(\lambda)$  defined in Eq. (13) is a boson pairing Hamiltonian.

For simplicity we subtract the ground state energy and write down  $\tilde{H}(\lambda) = H(\lambda) - E_0(\lambda)$ :

$$\begin{aligned} \tilde{H}(\lambda) &= \frac{1}{4} \left[ \frac{\hat{N}_d}{x_1 - \lambda} + \frac{\hat{N}_s}{x_2 - \lambda} \right]^2 - \frac{1}{2} \left[ \frac{\hat{N}_d}{(x_1 - \lambda)^2} - \frac{\hat{N}_s}{(x_2 - \lambda)^2} \right] \\ &\quad + W(\lambda) \left[ \frac{\hat{N}_d}{x_1 - \lambda} + \frac{\hat{N}_s}{x_2 - \lambda} \right] - S^+(\lambda) S^-(\lambda). \end{aligned} \quad (28)$$

Here  $\hat{N}_d$  and  $\hat{N}_s$  are total number operators for the *d* and *s* type bosons, i.e.

$$\hat{N}_d = \sum_{\mu} d_{\mu}^{\dagger} d_{\mu} \quad \text{and} \quad \hat{N}_s = s^{\dagger} s. \quad (29)$$

The last term in (28) is a pairing operator which corresponds to the  $P_6$  pairing operator of the *sd*-boson model. It involves the terms  $d_{\mu}^{\dagger} d_{\mu}^{\dagger} d_{\nu} d_{\nu}$  and  $s^{\dagger} s^{\dagger} s s$  as well as the cross terms  $d_{\mu}^{\dagger} d_{\mu}^{\dagger} s s$  and  $s^{\dagger} s^{\dagger} d_{\mu} d_{\mu}$ . Therefore  $\tilde{H}(\lambda)$  allows *s*-pairs to annihilate

each other to form  $s$ -pairs or  $d$ -pairs. Similarly  $d$ -pairs can also annihilate each other to produce  $s$  or  $d$  pairs.

Since  $\tilde{H}(\lambda) = H(\lambda) - E_0(\lambda)$ , energy eigenvalues of  $\tilde{H}(\lambda)$  can be obtained from those of  $H(\lambda)$  by subtracting the ground state energy. Eigenvalues of  $H(\lambda)$  were presented in the previous section. Gaudin's equation (19) tells us that the energy eigenvalues of  $\tilde{H}(\lambda)$  are

$$\tilde{E}_n(\lambda) = -2 \sum_{\alpha=1}^n \frac{W(\lambda) - W(\xi_\alpha)}{\lambda - \xi_\alpha}. \quad (30)$$

Here,  $\xi_1, \xi_2, \dots, \xi_n \in \mathbb{C}$  are obtained by solving the Bethe ansatz Eqs. (18).

Here we present a method to calculate these energy eigenvalues in an analytical way without solving the Bethe ansatz equations. Let us start with substituting  $W(\lambda)$  given in (27) into the energy formula (30) to find

$$\tilde{E}_n(\lambda) = -2 \sum_{\alpha=1}^n \left[ \frac{-s_1}{(x_1 - \lambda)(x_1 - \xi_\alpha)} + \frac{-s_2}{(x_2 - \lambda)(x_2 - \xi_\alpha)} \right]. \quad (31)$$

We see that it is not necessary to compute all the unknowns  $\xi_1, \xi_2, \dots, \xi_n \in \mathbb{C}$  one by one to evaluate  $\tilde{E}_n(\lambda)$ . It is sufficient to compute the following sums:

$$S_1 = \sum_{\alpha=1}^n \frac{1}{x_1 - \xi_\alpha} \quad S_2 = \sum_{\alpha=1}^n \frac{1}{x_2 - \xi_\alpha}. \quad (32)$$

We compute  $S_1$  and  $S_2$  using the symmetries of the equations of Bethe ansatz. It is easy to show that Eqs. (18) imply

$$\sum_{\alpha=1}^n W(\xi_\alpha) = 0 \quad \text{and} \quad \sum_{\alpha=1}^n \xi_\alpha W(\xi_\alpha) = \frac{n(n-1)}{2} \quad (33)$$

no matter what the form of  $W(\lambda)$  is. Substituting  $W(\lambda)$  given in (27) in above equalities one finds two equations in  $S_1$  and  $S_2$ . Solving these equations for  $S_1$  and  $S_2$  and then substituting them in equation (31) one finds<sup>10</sup>

$$\tilde{E}_n(\lambda) = \frac{1}{(x_1 - \lambda)(x_2 - \lambda)} \left[ n(n-1) + (n_1 + n_2) \frac{n}{2} \right]. \quad (34)$$

This is an analytical expression for all energy eigenvalues of  $\tilde{H}(\lambda)$ . It is computed without solving the Bethe ansatz equations. To find the corresponding energy eigenstates, however, one needs to solve the Bethe ansatz equations because each one of the quantities  $\xi_1, \xi_2, \dots, \xi_n \in \mathbb{C}$  is needed in order to write down the Bethe ansatz state in equation (17).

Here, we present a method to turn the problem of solving the Bethe ansatz equations for the  $n^{\text{th}}$  excited state into a problem of finding the roots of an  $n^{\text{th}}$  order polynomial. We substitute (27) in the equations of Bethe ansatz and then

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apply the change of variables  $\xi_\alpha = x_2 + \zeta_\alpha(x_1 - x_2)$ . Bethe ansatz equations then assume the following form:

$$\sum_{\beta \neq \alpha}^n \frac{1}{\zeta_\alpha - \zeta_\beta} + \frac{n_2/4}{\zeta_\alpha} - \frac{n_1/4}{1 - \zeta_\alpha} = 0 \quad \text{for } \alpha = 1, 2, \dots, n. \quad (35)$$

It was shown by Stieltjes<sup>11</sup> that when  $\zeta_\alpha$  obey (35), the polynomial

$$p_n(\zeta) = \prod_{\alpha=1}^n (\zeta - \zeta_\alpha) \quad (36)$$

satisfies the hypergeometric differential equation

$$\zeta(1 - \zeta)p_n''(\zeta) - \frac{n_1}{2}p_n'(\zeta) + n(n + \frac{n_1 + n_2}{2} - 1)p_n(\zeta) = 0. \quad (37)$$

This means that  $\zeta_\alpha$  for  $\alpha = 1, 2, \dots, n$  are the roots of the following polynomial:

$$\sum_{k=1}^{\infty} \frac{(-n)_k (n + n_1/2 + n_2/2 - 1)_k}{k! (n_2/2)_k} \zeta^k = 0. \quad (38)$$

$$\text{Here } (a)_0 = 0 \quad (39)$$

$$(a)_n = a(a+1) \cdots (a+n-1) \quad \text{for } n = 1, 2, \dots \quad (40)$$

The polynomial in (38) terminates when  $k = n$  because  $(-n)_{n+1} = 0$ . Consequently, finding the solution of the equations of Bethe ansatz for the  $n^{\text{th}}$  energy level is now reduced to the problem of finding the roots on an  $n^{\text{th}}$  order polynomial. In principle, we can find the roots of polynomials analytically up to the forth order. Therefore, first four excited energy eigenstates can be computed analytically. Solutions for the higher order eigenstates can be performed using numerical techniques.

#### 4. A New Solution to Gaudin's Equation

In this section we briefly mention another solution to Gaudin's equation given in (4). Instead of the constraint (i) given in Eq. (5) this new solution obeys  $w_{ij}^\alpha + w_{ji}^\alpha = 2q$  together with the other two constraints (ii) and (iii). The solution is given by<sup>12</sup>

$$w_{ij}^\alpha = q \coth[q(u_i - u_j)] + q \quad \text{for } \alpha = 1, 2, 3. \quad (41)$$

Here  $q$  is a complex parameter. In the limit where  $q \rightarrow 0$  this new solution approaches to the rational solution of Gaudin given by (6). For this reason we will denote the corresponding Gaudin magnet operators by  $h_i^{(q)}$ . In other words

$$h_i^{(q)} = \sum_{\substack{j=1 \\ j \neq i}}^N [q \coth[q(u_i - u_j)] + q] \vec{t}_i \cdot \vec{t}_j. \quad (42)$$

These operators mutually commute:  $[h_i^{(q)}, h_j^{(q)}] = 0$ . This Hamiltonian and its Bethe ansatz solution was discussed in Ref. 12.

### 5. A Connection with Calogero-Sutherland Model

Calogero model was first introduced in 1969 as a many-body system in one dimension with inverse square two-body interactions.<sup>13</sup> A variant of Calogero model introduced by B. Sutherland in 1971 includes two-body inverse sine square potentials.<sup>14</sup> Both models are prime examples of exactly solvable systems and have many interesting features. Originally both models involved scalar particles. But generalizations of these models to particles with spin degree of freedom which are called spin-Calogero and spin-Sutherland models also received increasing attention.<sup>15,16,17,18,19</sup> Spin-Calogero model Hamiltonian is given by

$$H^{(0)} = \sum_{i=1}^N \frac{1}{2} p_i^2 + g \sum_{\substack{i,j=1 \\ i < j}}^N \frac{\vec{t}_i \cdot \vec{t}_j}{(x_i - x_j)^2}. \quad (43)$$

This model describes non-relativistic particles which are free to move in one-dimension. Here  $x_i$  denote the positions,  $p_i$  denote the momenta and masses are scaled to unity.  $\vec{t}_i$  for  $i = 1, 2, \dots, N$  represent independent angular momenta of the particles.

Spin-Sutherland model, on the other hand, is given with the Hamiltonian

$$H^{(q)} = \frac{1}{2} \sum_{i=1}^N p_i^2 + q^2 g \sum_{\substack{i,j=1 \\ i < j}}^N \frac{\vec{t}_i \cdot \vec{t}_j}{\sinh^2[q(x_i - x_j)]}. \quad (44)$$

Here  $q$  is a complex parameter such that in the limit where  $q \rightarrow 0$ , the later model approaches to the former. It is possible to view Sutherland's model as a Calogero model on a circle, i.e. a system of particles which are constrained to move on the circle and interacting through Calogero potential along the cord distance between them. If we adopt this view then  $q = i\pi/d$  where  $d$  is the circumference of the circle.

One can write the spin-Calogero model Hamiltonian using the rational Gaudin magnet Hamiltonians given in Eq. (9) as

$$H^{(0)} = \frac{1}{2} \sum_{i=1}^N \left( p_i^2 + i \frac{g}{2} [p_i, h_i] \right). \quad (45)$$

Similarly, the spin-Sutherland model Hamiltonian can be written in the same form

$$H^{(q)} = \frac{1}{2} \sum_{i=1}^N \left( p_i^2 + i \frac{g}{2} [p_i, h_i^{(q)}] \right) \quad (46)$$

but this time we using the operators  $h_i^{(q)}$  which are given by Eq. (42).

This connection between Calogero-Sutherland model and the Gaudin algebras is closely related to the Lax formulation of Calogero-Sutherland model. To see this let us first substitute  $w_{ij}^1 = w_{ij}^2 = w_{ij}^3 = x(u_i - u_j)$  in Eq. 4. Here  $x(\xi)$  is a function which obeys  $x(\xi) + x(-\xi) = 2q$ . If we then take the derivative of Eq. 4 with respect to  $u_i$  and substitute  $\xi = u_j - u_i$  and  $\eta = u_i - u_k$  we find

$$x(\xi)x'(\eta) - x'(\xi)x(\eta) = x(\xi + \eta)[x'(\eta) - x'(\xi)].$$

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It is well known that when  $x(\xi)$  satisfies this equation, a many body system with the two-body potential  $v = -x'$  is a Lax type system. In other words, its equations of motion can be written in the form of a Lax pair (see Ref. 20 for a review). If we use the rational solution of Eq. 4 given in Eq. 6 then the two-body potential  $v = x'$  is equal to the Calogero potential. On the other hand, if one uses of the new solution given by Eq. 41 then  $v = -x'$  turn out to be the Sutherland potential.

## 6. Conclusion

In this paper we suggested a few possible directions to extend the method of Richardson and Gaudin. Results of Section 3 are very encouraging because we see that one can obtain analytical results without explicitly solving the equations of Bethe ansatz. The method of transforming the equations of Bethe ansatz into a problem of finding the roots of polynomials is also introduced. Finding the roots of polynomials is much more established problem and this trick may come handy in numerical calculations as well.

In this paper we relaxed one of the conditions imposed by Gaudin on the solutions of Eq. (4). This way we obtained a new and physically interesting solution. A new set of mutually commuting Hamiltonians  $h_i^{(q)}$  are thus identified. Evidently, these Hamiltonians can be related to spin-Sutherland model in a way which is parallel to the relation between the rational Gaudin magnet Hamiltonians and the spin-Calogero model. Application of algebraic Bethe ansatz technique to  $h_i^{(q)}$  yields a new Gaudin type algebra. This algebra admits a one parameter family of Hamiltonians  $H(\lambda)$  in a way which is analogous to other Gaudin algebras.

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## References

1. J. Bardeen, L.N. Cooper and J.R. Schrieffer, *Phys. Rev.* **108**, 1175 (1957).
2. R.W. Richardson, *Phys. Lett.* **B3**, 277 (1963).
3. R.W. Richardson, *J. Math. Phys.* **18**, 1802 (1977).
4. M. Gaudin, *J. Physique* **37**, 1087 (1976).
5. M. Gaudin, *La Fonction d'onde de Bethe*, Collection du Commissariat à l'énergie atomique, Masson, Paris, 1983.
6. L. Amico, A. Di Lorenzo and A. Osterloh, *Phys. Rev. Lett.* **86**, 5759 (2001) [arXiv:cond-mat/0105537]; J. Dukelsky, C. Esebbag and P. Schuck, *Phys. Rev. Lett.* **87**, 066403 (2001) [arXiv:cond-mat/0107477]; J. Dukelsky and S. Pittel, and G. Sierra, *Rev. Mod. Phys.* **76**, 643 (2004).

7. A. Arima and F. Iachello, *Annals Phys.* **99**, 253 (1976).
8. A. Arima and F. Iachello, *Phys. Rev. Lett.* **40**, 385 (1978).
9. A. B. Balantekin, arXiv:nucl-th/0312072.
10. A. B. Balantekin, T. Dereli and Y. Pehlivan, *J. Phys.* **G30**, 1225 (2004) [arXiv:nucl-th/0407006].
11. T.J. Stieltjes, 1914 *Sur Quelques Theoremes d'Algebre, Oeuvres Completes*, V. 11 (Groningen:Noordhoff).
12. A.B. Balantekin, T. Dereli, and Y. Pehlivan, to be submitted for publication.
13. F. Calogero, *J. Math. Phys.* **10**, 2191 (1969); **10** 2197 (1969); **12**, 419 (1971).
14. B. Sutherland, *Phys. Rev.* **A4**, 2019 (1971); **A5**, 1372 (1972).
15. J. Gibbons and T. Hermesen, *Physica* **D11**, 337 (1984).
16. S. Wojciechowski, *Phys. Lett.* **A111**, 101 (1985).
17. Z.N.C. Ha and F.D.M. Haldane, *Phys. Rev.* **B4**, 9359 (1992).
18. N. Kawakami, *Phys. Rev.* **B46**, 1005 (1992); **B46**, 3191 (1992).
19. J. A. Minahan and A. P. Polychronakos, *Phys. Lett.* **B302**, 265 (1993) [arXiv:hep-th/9206046]; **B326**, 288 (1994); A. P. Polychronakos, *Phys. Rev. Lett.* **89**, 126403 (2002) [arXiv:hep-th/0112141].
20. M.A. Olshanetsky and A.M. Perelomov, *Phys. Rep.* **71**, 313 (1981).